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PERTURBATION SOLUTIONS FOR WAVE PROPAGATION IN NONLINEARLY ELASTIC RODS

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Abstract—An asymptotic method for the problem of wave propagation in rods is applied to a nonlinearly elastic material obeying the neo-Hookean constitutive law. The displacement field is described by a perturbation series in the small radius of the rod. The first-order term corresponds to elementary theory, while subsequent terms give lateral inertia, shear and higher-order effects. First-order and second-order solutions are obtained for several values of the parameter characterizing the nonlinearity of the constitutive law.

1. INTRODUCTION

THE three-dimensional boundary-value problem of axisymmetric wave propagation in a circular elastic rod (not necessarily linear) can be attacked by the perturbation method [1]. The differential equations are expanded in series about the small parameter ε, representing the ratio of undeformed rod radius to some characteristic length related to the length of the pulse. The boundary conditions are obtained from a variational principle similar to that of Reissner [2].

The problem is thus reduced to a series of boundary-value problems, each representing a successively higher order of approximation. The first-order problem corresponds to elementary rod theory and, though nonlinear, can be solved analytically (in the form of a "simple-wave" solution) under certain conditions. The higher-order problems are linear, but with coefficients depending on the first-order solution; they are best solved numerically by the method of characteristics.

For sufficiently small ε, the second-order solution represents an adequate correction to the elementary theory. We shall present first-order and second-order solutions for both displacement and traction boundary-value problems and for several values of the parameter characterizing the nonlinearity of the material. We shall also show under what conditions the nonlinear problem may be approximated by the linear one.

2. STATEMENT OF THE PROBLEM

Consider a semi-infinite, uniform, nonlinearly elastic rod. Let the position of the material particles be defined at all times with respect to a fixed cylindrical coordinate system (R, Θ, Z) where R, Θ, Z define the position of the material particles at $t = 0$. For

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times $t > 0$, let us assume an axisymmetric motion of the rod in the form

$$r = r(R, Z, t), \tag{1}$$

$$\theta = \Theta, \tag{2}$$

$$z = z(R, Z, t). \tag{3}$$

As the stress measure let us use the first Piola–Kirchhoff stress T_{Ik} [3, p. 124]. Actually here, as the first Piola–Kirchhoff stress tensor we use the transpose of the one defined in [3, p. 124]. In our case $\mathbf{T} = J\mathbf{F}^{-1}\mathbf{T}_c$ where $\mathbf{T}_c$ the Cauchy stress tensor and $J = \det \mathbf{F}$.

As a nonlinear elastic constitutive law we chose that of the neo-Hookean material which is a special case of the Mooney–Rivlin constitutive theory for rubber [4]. The relation between the first Piola–Kirchhoff stress and the deformation gradient in this case is given by

$$\mathbf{T} = -p\mathbf{F}^{-1} + b\mathbf{F}^T, \tag{4}$$

where p is the pressure parameter and will be determined in the course of the solution, b is a material parameter which will be chosen to be $E/3$, where E is the Young modulus. With this value of b the neo-Hookean material reduces to the linear elastic material for small deformations. $\mathbf{F}$ is the deformation gradient whose physical components in the cylindrical coordinate system are:

$$\mathbf{F} = \begin{bmatrix} \dfrac{\partial r}{\partial R} & 0 & \dfrac{\partial r}{\partial Z} \\ 0 & \dfrac{r}{R} & 0 \\ \dfrac{\partial z}{\partial R} & 0 & \dfrac{\partial z}{\partial Z} \end{bmatrix}.$$

Due to the specific form of the above deformation gradient the stresses $T_{R\theta}$, $T_{\Theta r}$, $T_{\Theta z}$, $T_{Z\theta}$ will identically vanish and the other stresses will be independent of Θ.

The equation of motion in the first Piola–Kirchhoff stress for our axisymmetric system will be

$$\frac{\partial}{\partial R}T_{Rr} + \frac{\partial}{\partial Z}T_{Zr} + \frac{1}{R}(T_{Rr} - T_{\Theta\theta}) = \rho\ddot{r}, \tag{5}$$

$$\frac{\partial}{\partial R}T_{Rz} + \frac{\partial}{\partial Z}T_{Zz} + \frac{1}{R}T_{Rz} = \rho\ddot{z}. \tag{6}$$

Let us complete the definition of the problem with the description of boundary and initial values. Initial conditions are

$$r(R, Z, 0) = R, \qquad \dot{r}(R, Z, 0) = 0, \tag{7}$$

$$z(R, Z, 0) = Z, \qquad \dot{z}(R, Z, 0) = 0, \tag{8}$$

where

$$(\dot{\ }) = \frac{\partial}{\partial t}.$$

The boundary conditions on the lateral surface of the rod are

$$T_{Rr}|_{R=a} = 0, \qquad T_{Rz}|_{R=a} = 0. \tag{9}$$

Two kinds of boundary conditions will be considered at the end:

(a) Traction boundary conditions

$$T_{Zx}(R,0,t) = n(R,t), \tag{10}$$

$$T_{Zr}(R,0,t) = q(R,t). \tag{11}$$

(b) Displacement boundary conditions

$$z(R,0,t) = d(R,t), \tag{12}$$

$$r(R,0,t) = e(R,t). \tag{13}$$

Let us non-dimensionalize the variables as follows

$$\bar{Z} = \frac{Z}{L}, \qquad \bar{R} = \frac{R}{a}, \qquad \bar{t} = \frac{t}{T}, \qquad \bar{z} = \frac{z}{L},$$

$$\bar{r} = \frac{r}{a}, \qquad \bar{T}_{Ik} = \frac{T_{Ik}}{E}, \qquad \bar{p} = \frac{p}{E},$$

where T is a time parameter defined by the input disturbance at the end of the rod, a is the radius, E is Young's modulus; L is a characteristic length along the axis and is taken as $L = Tc_0$, where c_0 is the rod velocity given by $(E/\rho)^{\frac{1}{2}}$. Let us also define

$$\varepsilon = \frac{a}{L}.$$

For the simplicity in the following formulations let us omit the bars above the nondimensional variables. Any variable used from now on will be understood to be nondimensional. In the nondimensional variables equations (1)–(13) become

$$r = r(R,Z,t), \tag{14}$$

$$z = z(R,Z,t), \tag{15}$$

$$\frac{\partial}{\partial R}T_{Rr} + \frac{\partial}{\partial Z}T_{Zr}\varepsilon + \frac{1}{R}(T_{Rr} - T_{\Theta\theta}) = \varepsilon^2\ddot{r}, \tag{16}$$

$$\frac{\partial}{\partial R}T_{Rz} + \frac{\partial}{\partial Z}T_{Zz}\varepsilon + \frac{1}{R}T_{Rz} = \varepsilon\ddot{z}, \tag{17}$$

$$\mathbf{T} = -p\mathbf{F}^{-1} + \frac{b}{E}\mathbf{F}^T, \tag{18}$$

$$\mathbf{F} = \begin{bmatrix} \dfrac{\partial r}{\partial R} & 0 & \dfrac{\partial r}{\partial Z}\varepsilon \\ 0 & \dfrac{r}{R} & 0 \\ \dfrac{\partial z}{\partial Z}\dfrac{1}{\varepsilon} & 0 & \dfrac{\partial z}{\partial Z} \end{bmatrix}, \tag{19}$$

$$r(R, Z, 0) = R, \qquad \dot{r}(R, Z, 0) = 0, \tag{20a}$$

$$z(R, Z, 0) = Z, \qquad \dot{z}(R, Z, 0) = 0, \tag{20b}$$

$$T_{Rz}|_{R=1} = 0, \qquad T_{Rr}|_{R=1} = 0, \tag{21}$$

$$T_{Zz}(R, 0, t) = n(R, t), \tag{22}$$

$$T_{Zr}(R, 0, t) = q(R, t), \tag{23}$$

$$z(R, 0, t) = d(R, t), \tag{24}$$

$$r(R, 0, t) = e(R, t). \tag{25}$$

3. DERIVATION OF GOVERNING DIFFERENTIAL EQUATIONS

We now seek an asymptotic solution of the problem as follows. Expand z, r and p in the following asymptotic series

$$z = A + \varepsilon^2 B + \varepsilon^4 C + 0(\varepsilon^6), \tag{26}$$

$$r = M + \varepsilon^2 N + \varepsilon^4 L + 0(\varepsilon^6), \tag{27}$$

$$p = p_0 + \varepsilon^2 p_2 + \varepsilon^4 p_4 + 0(\varepsilon^6). \tag{28}$$

Since we are dealing with an incompressible material another expression which will be frequently used is the determinant of $\mathbf{F}$

$$\det \mathbf{F} = \frac{\partial r}{\partial R} \frac{r}{R} \frac{\partial z}{\partial Z} - \frac{\partial r}{\partial Z} \frac{r}{R} \frac{\partial z}{\partial R}. \tag{29}$$

The structure of A, B, M, N, p_0, p_2 will be assumed as follows

$$A = A(Z, t),$$

$$M = M_1(Z, t)R,$$

$$p_0 = p_0(Z, t),$$

$$B = B_1 + B_2 R^2,$$

$$N = N_1 R + N_2 R^3,$$

$$p_2 = p_{20} + p_{22} R^2.$$

Substituting expansions (26)–(28) in equations (16)–(19) and (21) and equating equal powers of ε to zero, we obtain the following results

$$M_1 = (A')^{-\frac{1}{2}}, \tag{30}$$

$$p_0 = \frac{b}{E}(A')^{-1}, \tag{31}$$

$$B_2 = \frac{A''}{4(A')^3}, \tag{32}$$

$$N_1 = \frac{-B_1'}{2(A')^{\frac{3}{2}}}, \tag{33}$$

$$N_2 = \frac{2(A'')^2 - A'A'''}{16(A')^{\frac{1}{2}}}, \tag{34}$$

$$p_{20} = \frac{b}{E}\,\frac{-3A'''A' + 8(A'')^2}{8(A')^6} - p_{22} - b\frac{B_1'}{E\,.\,(A')^2}, \tag{35}$$

$$p_{22} = \frac{b}{E}\,\frac{2A'A''' + 2(A'')^2 + 3(A'')^2(A')^3 - 12(A'')^2}{8(A')^2} - \frac{3}{8}\,\frac{(\dot{A}')^2}{(A')^3}, \tag{36}$$

where $(\dot{\ }) = \partial/\partial t, (\,') = \partial/\partial Z$.

The only independent functions up to $0(\varepsilon^3)$ are A and B_1. The equations governing A and B_1 are

$$A''\left[\frac{b}{E}\left(1 + \frac{2}{(A')^3}\right)\right] = \ddot{A}, \tag{37}$$

$$\ddot{B}_1 = B_1''\left[\frac{b}{E}\left(1 + \frac{2}{(A')^3}\right)\right] - \frac{6b}{E}\,\frac{A''}{(A')^4}B_1' + \eta(Z, t), \tag{38}$$

where

$$\eta(Z, t) = -\Psi'\frac{1}{A'} + \frac{1}{(A')^2}A''\Psi - 8\Omega\frac{b}{E} - \frac{2b}{E}\,\frac{N_2'}{(A')^{\frac{3}{2}}}$$

$$+\frac{b}{E}\,\frac{-A'A''A''' + 2(A'')^2}{16(A')^8} + \frac{A''}{(A')^2}p_{22}, \tag{39}$$

$$\Psi = \frac{b}{E}\,\frac{-3A'''A' + 8(A'')^2}{8(A')^6} - p_{22}, \tag{40}$$

and

$$\Omega = \frac{E}{16b}\left[\ddot{B}_2 - \frac{b}{E}B_2'' - \frac{b}{E}\,\frac{-A'A''A''' + 2(A'')^2}{4(A')^8} - \frac{1}{(A')^2}p_{22}A'' + \frac{p_{22}'}{A'} + \frac{2A''p_{22}}{(A')^2}\right]. \tag{41}$$

The stresses up to $0(\varepsilon^3)$ are

$$T_{Zz} = \frac{b}{E}\left(A' - \frac{1}{(A')^2}\right) + \varepsilon^2\left(-p_{20}(A')^{-1} - 2p_0M_1N_1 + \frac{b}{E}B_1'\right)$$

$$+\varepsilon^2R^2\left(-p_{22}\frac{1}{A'} - 4p_0M_1N_2 + \frac{b}{E}B_2'\right) + 0(\varepsilon^4), \tag{42}$$

$$T_{Rr} = \varepsilon^2\left(-\frac{1}{3}(A')^{\frac{1}{2}}B_1' - p_0A'N_1 - p_{20}A'M_1 + N_1\frac{b}{E}\right)$$

$$+\varepsilon^2R^2\left(-\frac{1}{3}(A')^{\frac{1}{2}}B_2' - p_0A'N_2 - p_{22}A'M_1 + 3N_2\frac{b}{E}\right) + 0(\varepsilon^4), \tag{43}$$

$$T_{\Theta\theta} = \varepsilon^2\left(-p_0B_1'M_1 - p_0N_1A' + 2p_0B_2M_1' - p_{20}M_1A' + N_1\frac{b}{E}\right)$$
$$+\varepsilon^2R^2\left(-p_0B_2'M_1 - 3p_0N_2A' - p_{22}M_1A' + N_2\frac{b}{E}\right) + 0(\varepsilon^4), \tag{44}$$

$$T_{Rz} = \varepsilon^3R^3\left(M_1'M_1p_{20} + p_0M_1'N_1 + p_0M_1N_1' + 2\xi\frac{b}{E}\right)$$
$$+\varepsilon^3R^3\left(M_1'M_1p_{22} + p_0M_1'N_2 + p_0N_2'M_1 + 4\Omega\frac{b}{E}\right) + 0(\varepsilon^4), \tag{45}$$

$$T_{Zr} = \varepsilon R\left(2p_0M_1B_2 + \frac{b}{E}M_1'\right) + \varepsilon^3\left(2B_2N_1p_0 + 2B_2p_{20}M_1 + 2\xi M_1p_0 + \frac{b}{E}N_1'\right)R$$
$$+\varepsilon^3R^3\left(2B_2N_2p_0 + 2B_2p_{21}M_1 + 4\Omega M_1p_0 + \frac{b}{E}N_2'\right) + 0(\varepsilon^4), \tag{46}$$

where

$$\xi = \frac{3E}{b}\left(-\frac{3b}{E}\Omega - (N_1+N_2)p_0M_1' - M_1p_0(N_1'+N_2') - M_1(p_{20}+p_{22})M_1'\right). \tag{47}$$

One can observe that

$$T_{Zr} \neq T_{Rz}.$$

Also a term $0(\varepsilon)$ is seen in (46), though no such a term occurs in the linear case. Thus shear effects are more important in the nonlinear material. The strains up to $0(\varepsilon^3)$ are

$$\epsilon_{Rr} = (1-M_1) + \tfrac{1}{2}(1-M_1)^2 - (N_1+3N_2R^2)(2-M_1)\varepsilon^2 + 0(\varepsilon^4) \tag{48}$$

$$\epsilon_{Zz} = (1-A') + \tfrac{1}{2}(1-A')^2 + \varepsilon^2(2-A')(B_1'+B_2'R^2) + 0(\varepsilon^4) \tag{49}$$

$$\epsilon_{\Theta\theta} = (1-M_1) + \tfrac{1}{2}(1-M_1)^2 + \varepsilon^2(2-M_1)(N_1+R^2N_2) + 0(\varepsilon^4) \tag{50}$$

$$2\epsilon_{Rz} = 2\epsilon_{Zr} = (M_1'R + (1-M_1)M_1'R)\varepsilon + \varepsilon^3[(N_1'R + N_2'R^3) + (1-M_1)(N_1'R + N_2R^3)$$
$$+(M_1R)(N_1+3R^2N_2) + 2B_2R(1-A') + 2B_2R] + 0(\varepsilon^5). \tag{51}$$

Here we have used the Green–St. Venant strain tensor

$$\mathbf{E} = \tfrac{1}{2}[\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T\mathbf{H}],$$

where

$$\mathbf{H} = \mathbf{F} - \mathbf{I} = \begin{bmatrix} \dfrac{\partial r}{\partial R} - 1 & 0 & \dfrac{\partial r}{\partial Z}\varepsilon \\ 0 & \dfrac{r}{R} - 1 & 0 \\ \dfrac{\partial z}{\partial R}\varepsilon & 0 & \dfrac{\partial z}{\partial Z} - 1 \end{bmatrix}.$$

At this point let us investigate under what restrictions on the deformation gradient the nonlinear solution of the problem reduces to the linear one. From (42) we have

$$T_{Zz}^{(0)} = \frac{b}{E}\left[A' - \frac{1}{(A')^2}\right].$$

If $A' \cong 1$, that is $A' = 1+e$ with $e \ll 1$, then we get

$$T_{Zz}^{(0)} = \frac{b}{E}\left[(1+e) - \frac{1}{(1+e)^2}\right] = \frac{b}{E}\left[1+e-(1-2e+0(e^2))\right] = \frac{b}{E}\left[3e+0(e^2)\right].$$

Neglecting $0(e^2)$ and choosing $b = E/3$ we obtain $T_{Zz} = e$. Here e is the infinitesimal longitudinal strain. For small deformations there is no distinction between the Cauchy and the first Piola–Kirchhoff stress and $T_{Zz} = \sigma_{zz}$. Thus we get $\sigma_{zz} = e$, which is the linear elastic constitutive law in nondimensional form. However, for the other equations such as (30)–(36) and others, such a reduction will not take place under $A' \cong 1$ alone, but the additional restrictions,

$$A'' \ll 1,\ A''' \ll 1 \qquad A'''' \ll 1 \qquad A' \ll 1$$

will be needed. Let us prove for example, that with the above conditions the $0(\varepsilon)$ term in T_{Zr} disappears for small deformations.

$$T_{Zr}^{(1)} = R\left[2p_0 M_1 B_2 + \frac{b}{E} M_1'\right],$$

also noting that

$$p_0 = \frac{b}{E}(A')^{-1}, \qquad B_2 = \frac{A''}{4(A')^3},$$

$$M_1' = (-\tfrac{1}{2})(A')^{-\frac{3}{2}}A'', \qquad M_1 = (A')^{-\frac{1}{2}},$$

we get

$$\begin{aligned} T_{Zr}^{(1)} &= R\left[\frac{b}{E}\frac{2A''}{4(A')^{\frac{1}{2}}(A')^3} + \frac{b}{E}\left(-\frac{1}{2}\right)A''(A')^{\frac{3}{2}}\right], \\ &= \frac{Rb}{2E}\left[\frac{A''}{(1+e)^{\frac{7}{2}}} - \frac{A''}{(1+e)^{\frac{3}{2}}}\right], \\ &= \frac{Rb}{2E}[A''(1-\tfrac{9}{2}e) - A''(1-\tfrac{3}{2}e)], \\ &= \frac{Rb}{2E}\,[-\tfrac{9}{2}A''e + \tfrac{3}{2}A''e]. \end{aligned}$$

But, noting that $A'' \ll 1$, we find $T_{Zr}^{(1)} \ll e$.

4. BOUNDARY CONDITIONS

The differential equations (37) and (38) can be solved if proper boundary conditions are imposed. In a linear problem instead of following the usual way of getting boundary conditions through matching with a boundary layer [5], we may use a variational principle [2]. Again, remembering that we are only interested in getting boundary conditions and not the already obtained equations of motion, we are going to use the so called first variational principle for hyperelastic materials [3, p. 326] and formally modify it for the dynamic case; namely, we shall form the Lagrangian function and formulate the variational principle in the form of Hamilton's principle.

4(a) *Traction boundary value problem*

First, let us apply the principle to the traction boundary value problem. Similar results can be obtained for the displacement boundary value problem.

Consider a deformation

$$\mathbf{x} = \mathbf{x}(\mathbf{X}), \tag{52}$$

where $\mathbf{X}$ denotes the vector of the original coordinates and $\mathbf{x}$ the deformation vector. Define the scalar function $W = W(\mathbf{F})$, (the strain energy), [3, p. 302] such that,

$$\mathbf{T}^T = \rho W_{,\mathbf{F}}.$$

Here $\mathbf{F}$ is the deformation gradient tensor, $\mathbf{T}$ is the first Piola–Kirchhoff stress tensor and ρ is the density in the undeformed state. For the neo-Hookean material W is given by

$$W(\mathbf{F}) = \tfrac{1}{2}b[I_B - 3],$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ and $I_\mathbf{B} = tr\,\mathbf{B}$ [3, p. 350]. The equations of motion are

$$\mathrm{Div}(\rho W_{,\mathbf{F}}) - \rho\ddot{\mathbf{x}} = 0 \quad \text{for all } \mathbf{X} \in V, \tag{53}$$

where Div is the divergence operator in the original coordinates and V is the original volume of the body. The traction boundary conditions are

$$\mathbf{T}^T . \mathbf{N} = \mathbf{t} \quad \text{for all } \mathbf{x} \in S. \tag{54}$$

Here S denotes the surface of the body, $\mathbf{N}$ is the normal in the original coordinates and $\mathbf{t}$ is the prescribed traction. Let us define the function (negative Lagrangian density)

$$\mathscr{F} = \rho W(F) - \tfrac{1}{2}\rho\dot{\mathbf{x}} . \dot{\mathbf{x}}.$$

Our variational principle (essentially Hamilton's principle) states, that, among all states of stress and deformation which satisfy equations (53), (54), the actually occurring state of one satisfying the variational equation

$$\delta \int_0^t \left[\int_V \mathscr{F}\, \mathrm{d}V - \int_S \mathbf{u} . \mathbf{t}\, \mathrm{d}S \right] \mathrm{d}t = 0. \tag{55}$$

After some manipulation equation (55) reduces to

$$\delta J(\mathbf{x}) = \int_V (\rho\ddot{x} - \mathrm{div}(\rho W_{,\mathbf{F}})) . \delta\mathbf{x}\, \mathrm{d}V + \int_S (\rho W_{,\mathbf{F}} . \mathbf{N} - \mathbf{t}) . \delta\mathbf{x}\, \mathrm{d}S. \tag{56}$$

Equation (56) is to be used in conjunction with the parametric expansions (26), (27) which satisfy the equations of motion and the boundary conditions for $R = 1$ identically in ε. This means that introduction of these expansions in (56) leaves us with a variational equation in the form

$$\int_{S_1} (\rho W_{,\mathbf{F}} \,.\, \mathbf{N} - \mathbf{t}) \,.\, \delta \mathbf{x} \, \mathrm{d}S = 0, \tag{57}$$

where S_1 denotes the surface of the end of the rod. Noting that $\mathbf{T}^T = \rho W_{,\mathbf{F}}$, expanding equation (57) and nondimensionalizing we get

$$\int_0^1 (T_{Zz} - n(R,t))\delta z R \, \mathrm{d}R + \int_0^1 (T_{Zr} - q(R,t))\delta r R \, \mathrm{d}R = 0.$$

Using

$$\delta z = \delta A + \varepsilon^2(\delta B_1 + R^2 \delta B_2) + 0(\varepsilon^4),$$

$$\delta r = \delta M + \varepsilon^2(R\delta N_1 + R^3 \delta N_2) + 0(\varepsilon^4),$$

and the proper expansions (42), (46) for T_{Zz} and T_{Zr} we get

$$\int_0^1 (T_{Zz}^{(0)} - n(R,t))R \, \mathrm{d}R = 0, \tag{58a}$$

$$\int_0^1 T_{Zz}^{(m)} R \, \mathrm{d}R = 0 \quad \text{for } m = 2, 4, 6, \ldots. \tag{58b}$$

4(b) *Displacement boundary-value problem*

For the displacement boundary-value problem the treatment is the same as in the preceding section. Instead of equation (55) we have now

$$\delta \int_0^t \left[\int_V \mathscr{F} \, \mathrm{d}V - \int_S (\mathbf{x} - \bar{\mathbf{x}}) \,.\, \mathbf{T}^T \mathbf{N} \, \mathrm{d}S \right] \mathrm{d}t = 0,$$

where $\bar{\mathbf{x}}$ is the prescribed deformation on the surface. Going through the same steps as in Section 1.b.2 of [1], corresponding to equation (57) one can obtain

$$\int_{S_1} (\mathbf{x} - \bar{\mathbf{x}}) \,.\, \delta \mathbf{T}^T \mathbf{N} \, \mathrm{d}S = 0.$$

Then proceeding along the same lines as in the traction boundary-value problem we get the following results:

$$\int_0^1 (z^{(0)} - \mathrm{d}(R,t))R \, \mathrm{d}R = 0, \tag{58c}$$

$$\int_0^1 z^{(m)} R \, \mathrm{d}R = 0 \qquad m = 2, 4, 6 \ldots. \tag{58d}$$

We obtain no boundary conditions either for the radial shear or for the radial displacement. They cannot be prescribed for the inner problem and remain as boundary layer effects.

5. SOLUTION OF THE TRACTION BOUNDARY-VALUE PROBLEM

The boundary condition (58a) gives

$$\int_0^1 (T_{Zz}^{(0)} - n(R,t))R\,\mathrm{d}R = 0. \tag{59}$$

Using equation (59) we get a boundary condition on A'.

$$\frac{b}{E}\left[A' - \frac{1}{(A')^2}\right] = 2\int_0^1 n(R,t)R\,\mathrm{d}R = N(t), \tag{60}$$

$$\frac{b}{E}\left[A' - \frac{1}{(A')^2}\right] = N(t). \tag{61}$$

From (20b) we get initial conditions

$$A'(z,0) = 1, \qquad \dot{A}(z,0) = 0. \tag{62}$$

The following equations determine completely the initial-boundary-value problem on A:

$$A''\left[\frac{b}{E}\left(1 + \frac{2}{(A')^3}\right)\right] = \ddot{A}, \tag{63}$$

$$\frac{b}{E}\left[A'(0,t) - \frac{1}{(A'(0,t))^2}\right] = N(t), \tag{64}$$

$$A'(z,0) = 1, \qquad \dot{A}(z,0) = 0. \tag{65}$$

For $m = 2$ (58b) gives

$$\int_0^1 T_{Zz}^{(2)} R\,\mathrm{d}R = 0. \tag{66}$$

But from (42) we get

$$T_{Zz}^{(2)} = T_{Zz}^{(20)} + T_{Zz}^{(22)} R^2,$$

where

$$T_{Zz}^{(20)} = \left[-\frac{1}{A'}\left(\Psi - \frac{b}{E}\frac{B_1'}{(A')^2}\right) - 2p_0 M_1 N_1 + \frac{b}{E}B_1'\right], \tag{67}$$

$$T_{Zz}^{(22)} = \frac{b}{E}B_1'\left[\frac{1}{(A')^3} + 1\right] - \frac{\Psi}{A'} - 2p_0 M_1 N_1, \tag{68}$$

and where Ψ was previously defined in (40). Denoting

$$L = -\frac{\Psi}{A'} - 2p_0 M_1 N_1 = L(A', A'', A'''), \tag{69}$$

$$W = T_{Zz}^{(22)}, \qquad W = W(A', A'', A''', \dot{A}'), \tag{70}$$

one obtains from (66)

$$\frac{b}{E}B_1'\left[\frac{1}{(A')^3} + 1\right]_{Z=0} + L(A', A'', A''')|_{Z=0} + \tfrac{1}{2}W(A', A'', A''', \dot{A}')|_{Z=0} = 0.$$

This furnishes a linear boundary condition on B_1'. Again, from equation (20b) we get the initial conditions on B_1. The following equations determine completely the initial-boundary-value problem on B_1.

$$\ddot{B}_1 = B_1''\left[\frac{b}{E}\left(1+\frac{2}{(A')^3}\right)\right] - \frac{6b}{E}\frac{A''}{(A')^4}B_1' + \eta(z, t), \tag{71}$$

$$\frac{b}{E}B_1'(0, t)\left[\frac{1}{(A'(0, t))^3}+1\right] + L(A', A'', A''')|_{Z=0} + \tfrac{1}{2}W(A', A'', A''', \dot{A}')|_{Z=0} = 0, \tag{72}$$

$$B_1'(z, 0) = 0, \qquad \dot{B}_1(z, 0) = 0. \tag{73}$$

5(a) *First-order solution*

Let us now define $s = A'$, $v = \dot{A}$; then equation (63) can be written as

$$s'\left[\frac{b}{E}\left(1+\frac{2}{s^3}\right)\right] = \dot{v}. \tag{74}$$

We also have the compatibility equation

$$\dot{s} = v'. \tag{75}$$

We denote $s(0, t)$ by $\tilde{s}(t)$. Equation (64) becomes symbolically

$$f(\tilde{s}) = N(t), \tag{76}$$

where

$$f(\tilde{s}) = \frac{b}{E}\left[\tilde{s} - \frac{1}{(\tilde{s})^2}\right].$$

Defining

$$g(k) = f^{-1}(k) \tag{77a}$$

and

$$h(t) = g(N(t)), \tag{77b}$$

we have

$$s(0, t) = h(t). \tag{78}$$

As initial conditions we have

$$s(z, 0) = 1, \qquad v(z, 0) = 0.$$

A general solution of equations (74) and (75) can be found by the method of characteristics. Introduce the characteristic variables α and β such that

$$\mathrm{d}\alpha = 0 \quad \text{on} \quad \frac{\mathrm{d}Z}{\mathrm{d}t} = c, \tag{79}$$

$$\mathrm{d}\beta = 0 \quad \text{on} \quad \frac{\mathrm{d}Z}{\mathrm{d}t} = -c, \tag{80}$$

where

$$c = \left[\frac{b}{E}\left(1+\frac{2}{s^3}\right)\right]^{\frac{1}{2}}.$$

Casting the equations in terms of these variables one can prove [6, pp. 65–71] that

$$c\frac{\partial s}{\partial \beta}-\frac{\partial v}{\partial \beta} = 0,$$

$$c\frac{\partial s}{\partial \alpha}+\frac{\partial v}{\partial \alpha} = 0,$$

which gives

$$\int_0^s c(\xi)\,\mathrm{d}\xi - v = K_2(\alpha), \tag{81}$$

$$\int_0^s c(\xi)\,\mathrm{d}\xi + v = K_1(\beta). \tag{82}$$

Let us label the characteristics such that $\alpha = \beta = t$ at $Z = 0$. The boundary conditions in terms of α and β will be

$$s = h(\alpha) \quad \text{on} \quad \alpha = \beta = t, \tag{83}$$

$$s = 1, v = 0 \quad \text{on} \quad \alpha = 0. \tag{84}$$

The wave front is given by $\alpha = 0$. Using equations (81), (82) and boundary conditions (83) and (84) we get for s and v

$$s = h(\alpha), \quad \text{where} \quad \alpha = t-\frac{Z}{c(s)}, \tag{85}$$

and

$$v = -\int_1^s c(\xi)\,\mathrm{d}\xi. \tag{86}$$

Suppose that $N(t)$ is given in equation (76); one can solve this nonlinear algebraic equation for $\tilde{s}$ and $h(t)$ is obtained numerically. At any value of t at $Z = 0$, $h(t)$ is known, that is, $\tilde{s}$ is known. At $Z = 0$, $t = t_1$, say, $s(0, t_1) = s^*$. Then the slope of the α characteristic issuing from $(0, t_1)$ is

$$\frac{\mathrm{d}t}{\mathrm{d}Z} = \frac{1}{c(s^*)},$$

and along the characteristic $t_1 = t - Z/c(s^*)$, the value of s is S^*. On the other hand from equation (86) the value of v on $(0, t_1)$ is

$$v^* = -\int_1^{s^*} c(\xi)\,\mathrm{d}\xi.$$

Knowing v^* we know that on the characteristic

$$t_1 = t - \frac{Z}{c(s^*)},$$

the value of v will be v^*. Thus from the implicit nature of equation (85) at any point (z, t) we cannot trace back the α characteristic and see what value of S it carries, but we can determine to what points in the (Z, t) plane a given value of s at $Z = 0$ will propagate. We will choose the mesh points in the (Z, t) plane and determine values of s at these points. The mesh points will be chosen as follows; the initial line $Z = 0$ will be divided by points Δt apart from which the α characteristics will emerge. These characteristics will be intersected by vertical lines ΔZ apart and the mesh points will be determined. Of course, for the above solution to be valid no shocks must form. This condition will impose on $h(t)$ a special dependence on time. This condition will be derived as follows. From equation (85),

$$s = A' = h(\alpha),$$

where

$$\alpha = t - \frac{Z}{c(s)}.$$

For no shocks to occur, $\dot{A}'$ must be finite for all values of Z and t. $\dot{A}'$ is given by

$$\dot{A}' = \frac{h_{,\alpha}}{1 - h_{,\alpha}\alpha_{,A'}}, \quad \text{where } \alpha_{,A'} = \frac{\partial \alpha}{\partial A'}.$$

For $\dot{A}'$ to be finite for all Z and t, we need

$$1 - h_{,\alpha}\alpha_{,A'} \neq 0.$$

If $h_{,\alpha}\alpha_{,A'} \leq 0$ this is clearly satisfied. But,

$$\alpha_{,A'} = \frac{Z}{2} \cdot \frac{1}{c(A')^2} \cdot \frac{\partial c}{\partial A'}.$$

Then for $Z > 0$ the sign of $\alpha_{,A'}$ is the same as the sign of $\partial c/\partial A'$. But in our case,

$$\frac{\partial c}{\partial A'} = -\frac{6}{(A')^2} \frac{b}{E} \cdot \frac{1}{2} \cdot \left(\frac{b}{E}\left(1 + \frac{2}{s^3}\right)\right)^{-1/2}$$

which implies $\alpha_{,A'} < 0$. Then we need to have $h_{,\alpha} \geq 0$. In short, the above solution is valid for a neo-Hookean material if

$$\frac{\partial h}{\partial t} \geq 0 \quad \text{for } t > 0.$$

This was a condition on $A'(0, t)$, but supposing that in the traction boundary-value problem the stress $N(t)$ is prescribed. Then one would have to know what the condition on $N(t)$ is. $N(t)$ being prescribed, for $A'(0, t)$ to be monotonically nondecreasing we need

$$\frac{b}{E}\dot{A}'\left[1 + \frac{2}{(A')^3}\right] = \dot{N}(t).$$

Then we get

$$\dot{A}' = \frac{\dot{N}(t)}{(b/E)(1+2/(A')^3)}. \tag{87}$$

Thus for $A' > 0$, if $N(t)$ is monotonically nondecreasing then $\dot{A}'(0, t)$ will be monotonically nondecreasing and no shocks form.

5(b) *Second-order solution*

Let us take equation (71) which governs B_1. As is seen, this is a linear non-homogeneous equation. This turns out to be the fact which makes the determination of all higher-order terms possible. The perturbation scheme on the nonlinear three-dimensional field equations, give, first, a homogeneous nonlinear equation together with linear equations governing the higher-order terms. It is remarkable that all these equations possess the same characteristic curves, a fact which facilitates the solution to a large extent. A numerical procedure will be used along the characteristics, using the mesh points as previously defined.

At this point if one analyzes equation (71) it will be seen that it involves the higher derivatives of A', these being A'', A''', A'''', $\dot{A}'$. For the numerical procedure a knowledge of these derivatives at the defined mesh points is necessary. They will be obtained as follows. Denote again A' by s. Let us find s'. From equation (85) differentiating with respect to Z we obtain

$$s' = \frac{\partial h}{\partial \alpha}[\alpha' + \alpha_{,s} s'], \tag{88}$$

and

$$s' = \frac{(\partial h/\partial \alpha)\alpha'}{1-(\partial h/\partial \alpha)\alpha_{,s}},$$

where

$$\alpha = t - \frac{Z}{c(s)}, \tag{89}$$

$$\alpha' = -\frac{1}{c(s)}, \tag{90}$$

$$\alpha_{,s} = \frac{Z(\partial c/\partial s)}{(c(s))^2}. \tag{91}$$

If, at any point (Z_0, t_0), s is known, then α' and $\alpha_{,s}$ are determined. $\partial h/\partial \alpha$ is given by

$$\frac{\partial h}{\partial \alpha} = \left.\frac{\partial h}{\partial t}\right|_{t=\alpha},$$

with

$$\frac{\partial h}{\partial t} = \frac{\partial g}{\partial k} \cdot \frac{\partial N}{\partial t}. \tag{92}$$

But from the inverse function theorem [7, p. 256] we have

$$\frac{\partial g}{\partial k} = \frac{1}{\partial f/\partial \tilde{s}},$$

and

$$\frac{\partial h}{\partial t} = \frac{1}{\partial f/\partial \tilde{s}} \cdot \frac{\partial N}{\partial t}.$$

Then we have

$$\frac{\partial h}{\partial \alpha} = \left.\frac{\partial h}{\partial t}\right|_{t=\alpha} = \frac{1}{\partial f/\partial \tilde{s}|_{\tilde{s}=\tilde{s}^*}} \cdot \left.\frac{\partial N}{\partial t}\right|_{t=\alpha},$$

where $s^* = h(\alpha)$, namely, if we are seeking $\partial h/\partial \alpha$ at a point (Z_0, t_0) on an α characteristic issuing at $Z = 0$, from $\alpha = t = t^*$, then we have

$$\tilde{s}^* = h(t^*),$$

$$\alpha_0 = t_0 - \frac{Z_0}{c(\tilde{s}^*)},$$

and $\partial h/\partial \alpha$ is clearly obtained.

Let us now find s''. Differentiating equation (88) with respect to Z, we obtain

$$s'' = \frac{\partial^2 h}{\partial \alpha^2}[\alpha' + \alpha_s s']^2 + [\alpha'' + \alpha_s' s' + \alpha_{,ss}(s')^2 + \alpha'_{,s} s' + \alpha_{,s} s''] \cdot \frac{\partial h}{\partial \alpha}. \tag{93}$$

In the above formula everything besides $\partial^2 h/\partial \alpha^2$ is readily obtained. $\partial^2 h/\partial \alpha^2$ is given by

$$\frac{\partial^2 h}{\partial \alpha^2} = \left.\frac{\partial^2 h}{\partial t^2}\right|_{t=\alpha}.$$

$\partial^2 h/\partial t^2$ can be obtained by differentiating (92) with respect to t:

$$\frac{\partial^2 h}{\partial t^2} = \left[\frac{\partial^2 N}{\partial t^2} \cdot \frac{\partial f}{\partial \tilde{s}} - \frac{\partial^2 f}{\partial \tilde{s}^2}\frac{\partial N}{\partial t}\frac{\partial \tilde{s}}{\partial t}\right] \cdot \frac{1}{(\partial f/\partial \tilde{s})^2},$$

then

$$\frac{\partial^2 h}{\partial t^2} = \left[\frac{\partial^2 N}{\partial t^2} \cdot \frac{\partial f}{\partial \tilde{s}} - \frac{\partial^2 f}{\partial \tilde{s}^2}\left(\frac{\partial N}{\partial t}\right)^2 \frac{1}{\partial f/\partial \tilde{s}}\right] \frac{1}{(\partial f/\partial \tilde{s})^2}, \tag{94}$$

$\partial^2 h/\partial \alpha^2$ becomes

$$\frac{\partial^2 h}{\partial \alpha^2} = \left.\frac{\partial^2 h}{\partial t^2}\right|_{t=\alpha} = \left[\left.\frac{\partial^2 N}{\partial t^2}\right|_{t=\alpha} \cdot \left.\frac{\partial f}{\partial \tilde{s}}\right|_{\tilde{s}=\tilde{s}^*} - \left.\frac{\partial^2 f}{\partial \tilde{s}^2}\right|_{\tilde{s}=\tilde{s}^*} \cdot \left(\left.\frac{\partial N}{\partial t}\right|_{t=\alpha}\right)^2 \cdot \frac{1}{(\partial f/\partial \tilde{s})|_{\tilde{s}=\tilde{s}^*}}\right]$$

$$\times \frac{1}{[(\partial f/\partial \tilde{s})|_{\tilde{s}=\tilde{s}^*}]^2}, \tag{95}$$

where $\tilde{s}^*$ was previously defined. Similarly s''' can be obtained by differentiating equation (93) with respect to Z. It will involve $\partial^3 h/\partial \alpha^3$ which is given by

$$\frac{\partial^3 h}{\partial \alpha^3} = \left.\frac{\partial^3 h}{\partial t^3}\right|_{t=\alpha}.$$

$\partial^3 h/\partial t^3$ can be obtained by differentiating $\partial^2 h/\partial t^2$ (94) with respect to t:

$$\frac{\partial^3 h}{\partial t^3} = \frac{(\partial^3 N/\partial t^3).(\partial f/\partial \tilde{s}) - (\partial^2 f/\partial \tilde{s}^2)(\partial \tilde{s}/\partial t)(\partial^2 N/\partial t^2)}{(\partial f/\partial \tilde{s})^2}$$
$$-\frac{(\partial^3 f/\partial \tilde{s}^3)(\partial \tilde{s}/\partial t)(\partial N/\partial t)^2 - 2(\partial N/\partial t)(\partial^2 N/\partial t^2)(\partial^2 f/\partial \tilde{s}^2)}{(\partial f/\partial \tilde{s})^3}$$
$$+\left[3\left(\frac{\partial f}{\partial \tilde{s}}\right)^2\left(\frac{\partial^2 f}{\partial \tilde{s}^2}\right)\left(\frac{\partial \tilde{s}}{\partial t}\right)\left(\frac{\partial^2 f}{\partial \tilde{s}^2}\right)\left(\frac{\partial N}{\partial t}\right)^2\right]\frac{1}{(\partial f/\partial \tilde{s})^6}, \tag{96}$$

where

$$\frac{\partial \tilde{s}}{\partial t} = \frac{1}{\partial f/\partial \tilde{s}} \cdot \frac{\partial N}{\partial t}.$$

We also need $\dot{A}'$:

$$\dot{A}' = \dot{s}, \quad \text{but } s = h(\alpha),$$

and

$$\dot{s} = h_{,\alpha}(\dot{\alpha} + \alpha_{,s}\dot{s}).$$

This will be readily determined since we already know $h_{,\alpha}$, therefore the only equation we need to solve numerically is the implicit equation $f(\tilde{s}) = N(t)$. The derivatives of s are obtained with no difficulty and one has all the necessary information to solve the equation for B_1. Let us put equation (71) in the form

$$\ddot{B}_1 = \phi B_1'' + \chi B_1' + \eta, \tag{97}$$

where

$$\phi = \frac{b}{E}\left(1 + \frac{2}{(A')^3}\right),$$
$$\chi = -\frac{6b}{E}\left[\frac{A''}{(A')^4}\right],$$

and η is as defined previously in equation (39). Let

$$\dot{B}_1 = v_1,$$
$$B_1' = s_1;$$

then equation (97) becomes

$$\dot{v}_1 = \phi s_1' + \chi s_1 + \eta.$$

We also have the compatibility equation: $v_1' = \dot{s}_1$. The boundary and initial conditions are

$$s_1(0,t)\cdot \frac{b}{E}\left[\frac{1}{(A')^3} + 1\right]_{Z=0} + L\Big|_{Z=0} + \tfrac{1}{2}W\Big|_{Z=0} = 0,$$
$$s_1(0,Z) = v(0,Z) = 0.$$

Noting that

$$\frac{\mathrm{d}v_1}{\mathrm{d}t} = \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial Z}\frac{\mathrm{d}Z}{\mathrm{d}t}, \tag{98}$$

we see that along the α characteristics (98) becomes

$$\frac{\mathrm{d}v_1}{\mathrm{d}t} = \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial Z}\phi^{\frac{1}{2}}.$$

Thus

$$\frac{\mathrm{d}v_1}{\mathrm{d}t} = \phi s_1' + \chi s_1 + \eta + \phi^{\frac{1}{2}}\dot{s},$$

$$\frac{\mathrm{d}v_1}{\mathrm{d}t} = \phi^{\frac{1}{2}}(\dot{s}_1 + \phi^{\frac{1}{2}}s_1') + \chi s_1 + \eta. \tag{99}$$

Along the α characteristics we have

$$\frac{\mathrm{d}v_1}{\mathrm{d}t} = \phi^{\frac{1}{2}}\frac{\mathrm{d}s_1}{\mathrm{d}t} + \chi s_1 + \eta. \tag{100a}$$

Similarly along the β characteristics we have

$$\frac{\mathrm{d}v_1}{\mathrm{d}t} = -\phi^{\frac{1}{2}}\frac{\mathrm{d}s_1}{\mathrm{d}t} + \chi s_1 + \eta. \tag{100b}$$

We will base our numerical procedure on equations (100a) and (100b) and approximate them by finite differences. Assuming that the values of the variables have been obtained on an α characteristic line, the values on the next characteristic line will be obtained (Fig. 1). The reason for proceeding thus and not along horizontal lines is that the mesh points are determined beforehand and not along with the numerical procedure. Suppose

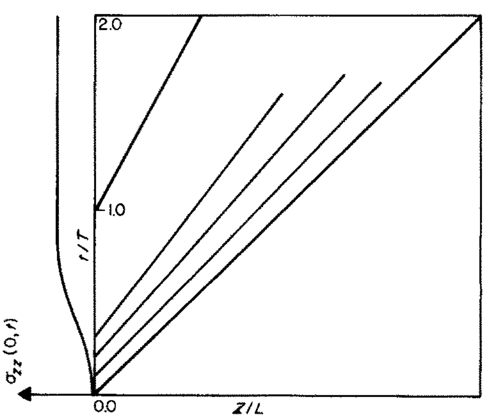

FIG. 1. Stress input and α characteristics.

that v_1 and s_1 have been obtained along the characteristic χ_1. At the point P on the characteristic α_2, (Fig. 2) equations (100a) and (100b) give

$$\frac{v_1(P)-v_1(Q)}{m} = (\phi(P))^{\frac{1}{2}}\frac{s_1(P)-s_1(Q)}{m}+\chi(P)s_1(P)+\eta(P),$$

$$\frac{v_1(P)-v_1(Q')}{l} = -(\phi(P))^{\frac{1}{2}}\frac{s_1(P)-s_1(Q')}{l}+\chi(P)s_1(P)+\eta(P).$$

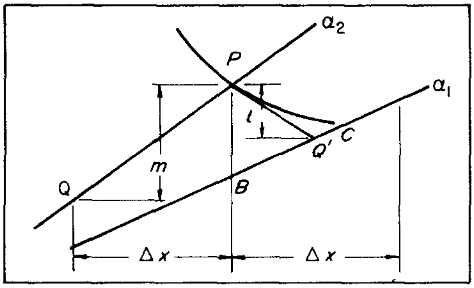

FIG. 2. Diagram for difference scheme.

Here we are approximating the β characteristic through P by its tangent. We don't know the values of v_1 and s_1 at Q' but their values at B and C are already known. We can use interpolation between B and C to determine the values at Q'. Linear interpolation will be used:

$$v_1(Q') = v_1(B)+\frac{(v_1(C)-v_1(B))}{\overline{BC}}\times\overline{Q'B},$$

$$s_1(Q') = s_1(B)+\frac{(s_1(C)-s_1(B))}{\overline{BC}}\times\overline{Q'B}.$$

The determination of B_1 is thus completed. (Refer to Fig. 2 for symbols used above.)

5(c) *Example of the traction boundary value problem*

An example is worked out for the traction boundary-value problem. Keeping in mind that the nondimensionalization was made with respect to E (Young's modulus), T (rise time of the pulse) and C_0T (where C_0 is the elastic rod velocity), it is possible to work throughout the example with nondimensional quantities.

To avoid the solution of the nonlinear algebraic equation at the boundary, a very specific input is chosen at the end, as follows

$$T_{Zz}(0,t) = \frac{1}{3}\left[\left(1+m\sin^3\frac{\pi}{2}t\right)-\frac{1}{(1+m\sin^3(\pi/2)t)^3}\right] \quad \text{for } 0 \le t \le 1, \tag{101a}$$

$$T_{Zz}(0,t) = \frac{1}{3}\frac{(1+m)^4-1}{(1+m)^3} \quad \text{for } t \ge 1. \tag{101b}$$

Such a special variation on time is chosen to secure the non-decreasing character of the input. In the above formula m is a parameter which can be varied at will according to the magnitude of the deformations. For $m = 0{\cdot}001$ we will have small deformations and the results may be expected to coincide with the linear elastic theory, According to equation (60)

$$N(t) = 2\int_0^1 T_{Zz}R\,\mathrm{d}R,$$
$$= \frac{1}{3}\left[\left(1+m\sin^3\frac{\pi}{2}t\right)-\frac{1}{(1+m\sin^3(\pi/2)t)^3}\right].$$

From equation (64) we get

$$A'(0,t) = 1+m\sin^3\frac{\pi}{2}t.$$

To be able to make a true comparison between the input stress and the stress at a later station let us define

$$T^*_{Zz} = 2\int_0^1 T_{Zz}R\,\mathrm{d}R.$$

Then we will have

$$T^*_{Zz} = T^{(0)}_{Zz}+\varepsilon^2[T^{(20)}_{Zz}+T^{(22)}_{Zz}\tfrac{1}{2}],$$
$$T^*_{Zz} = T^{(0)}_{Zz}+\varepsilon^2 T^{(2)*}_{Zz}.$$

At different values of Z along the rod we shall be interested in the response for values of t bounded by the characteristics issuing from $t = 0$ and $t = 1{\cdot}0$ (Fig. 1). Our purpose will be to analyze the importance of the second order effect for different values of m and compare it with the linear case. The values of m used are: $m = 0{\cdot}001$ (Fig. 3), $m = 0{\cdot}01$ (Fig. 4), $m = 0{\cdot}1$ (Fig. 5), $m = 1{\cdot}0$ (Fig. 6), $m = 2{\cdot}0$ (Fig. 7), $m = 3{\cdot}0$ (Fig. 8).

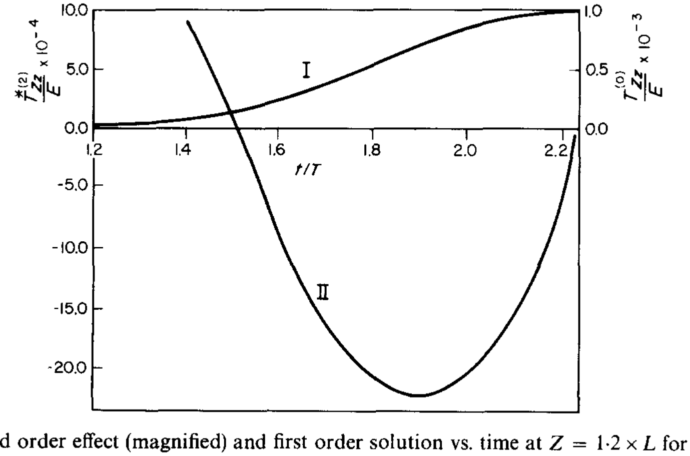

FIG. 3. Second order effect (magnified) and first order solution vs. time at $Z = 1{\cdot}2\times L$ for $m = 0{\cdot}001$.

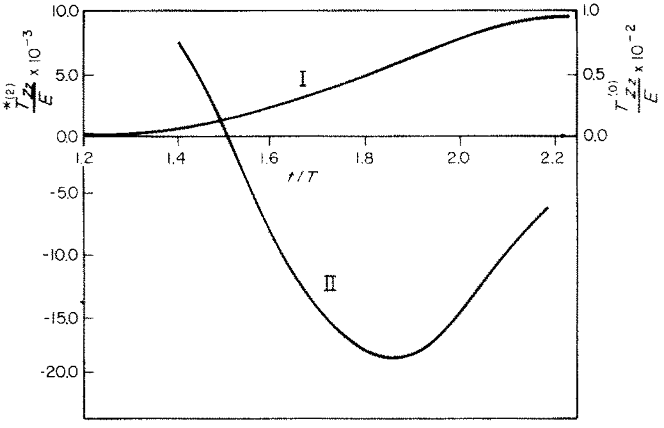

FIG. 4. Second order effect (magnified) and first order solution vs. time at $Z = 1{\cdot}2 \times L$ for $m = 0{\cdot}01$.

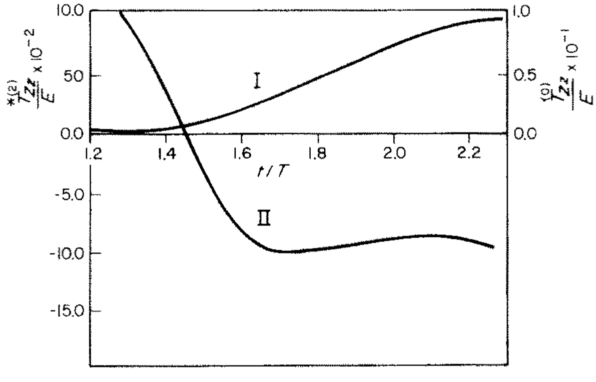

FIG. 5. Second order effect (magnified) and first order solution vs. time at $Z = 1{\cdot}2 \times L$ for $m = 0{\cdot}1$.

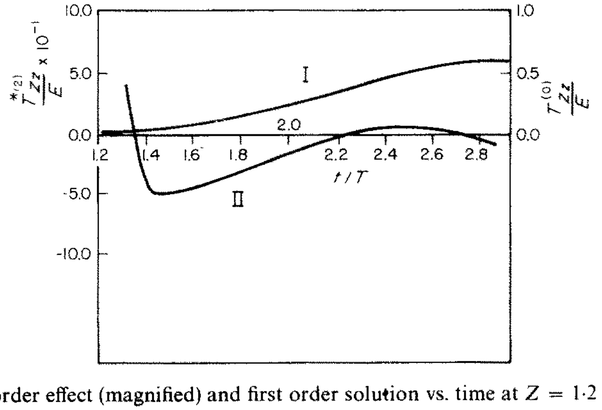

FIG. 6. Second order effect (magnified) and first order solution vs. time at $Z = 1{\cdot}2 \times L$ for $m = 1{\cdot}0$.

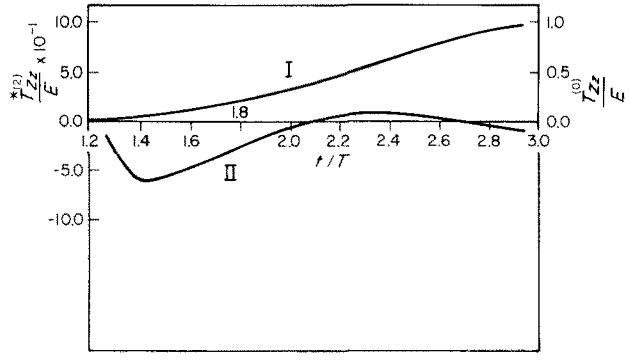

FIG. 7. Second order effect (magnified) and first order solution vs. time at $Z = 1{\cdot}2 \times L$ for $m = 2{\cdot}0$.

In all cases the results are evaluated at a distance $Z = 1{\cdot}2$ (nondimensional) from the end of the rod. Since Figs. 3–8 contain similar notations let us look at Fig. 3 and explain the several elements in it. On the abscissa the time elapsed since the arrival of the wave at the station $Z = 1{\cdot}2$ is recorded. On the right hand side of the graph the vertical axis denotes the first order solution. On the left hand side of the graph the vertical axis denotes the second-order solution $T_{Zz}^{(2)*}$. To make a clearer comparison we have not multiplied $T_{Zz}^{(2)*}$ by ε^2. The actual second-order term will be obtained by multiplying the graph of $T_{Zz}^{(2)*}$ by ε^2. Graph I represents the first-order solution: graph II represents the second-order solution. We are interested only in the second-order effect around the peak, since in regions where it predominates over the first-order solution validity is not to be sought. For $m = 0{\cdot}001$ the solution reduces to the linear one [1]. The second-order effect in Fig. 3 is exactly like the one described in [1]. For increasing values of m the first-order solution $T_{Zz}^{(0)}$ does not change qualitatively. The magnitude of the peak is the same as the one of the input at the end of the rod. There occurs, however, a spreading of the pulse over a larger interval of time. That is, a decrease in the strain rate is observed. For example for $m = 0{\cdot}001$ at $Z = 1{\cdot}2$ the peak of the first-order solution is reached in $\Delta t = 1$ unit (nondimensional) after the arrival of the wave. This is to be expected since the peak in the

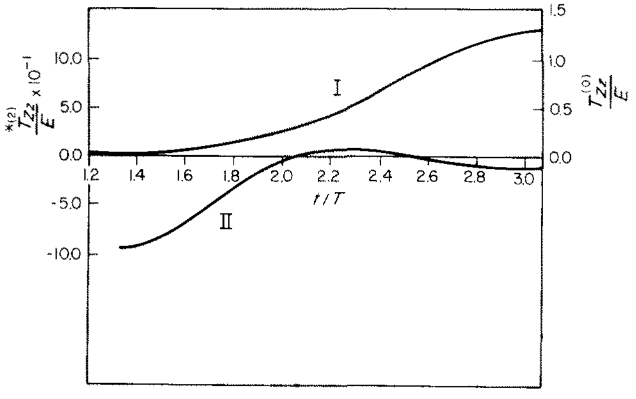

FIG. 8. Second order effect (magnified) and first order solution vs. time at $Z = 1{\cdot}2 \times L$ for $m = 3{\cdot}0$.

input is reached in $\Delta t = 1$ unit also and no spreading of the pulse is expected in the nearly linear case. For $m = 3$ this value at $Z = 1{\cdot}2$ becomes $\Delta t = 1{\cdot}9$.

As to the second-order effect, for $m = 0{\cdot}001$ it reduces to the one in the linear case. There is no correction in this case to the magnitude of the peak with our specific input and the most significant corrections occur on the *sides* of the pulse. Again, restricting our attention only around the peak, a considerable decrease of the second-order effect is observed for higher values of m. In Figs. 7 and 8 this correction becomes nearly negligible. It is of interest to notice, though, that for very small deviations from the linear case, the second-order effect changes rapidly.

6. SOLUTION OF THE DISPLACEMENT BOUNDARY-VALUE PROBLEM

A brief description of the solution will be given since it follows almost the same lines of solution of the traction boundary-value problem. Equation (58c) when proper expressions are substituted into it gives

$$A(0, t) = 2\int_0^1 \mathrm{d}(R, t)R\,\mathrm{d}R = D(t), \tag{102}$$

$$A(0, t) = D(t). \tag{103}$$

Differentiating with respect to time, we obtain

$$\dot{A}(0, t) = \dot{D}(t) = V(t). \tag{104}$$

The following equations determine completely the initial boundary value problem on A

$$A''\left[\frac{b}{E}\left(1+\frac{2}{(A')^3}\right)\right] = \ddot{A}, \tag{105}$$

$$\dot{A}(0, t) = V(t), \tag{106}$$

$$A'(Z, 0) = 1 \qquad \dot{A}(Z, 0) = 0. \tag{107}$$

Similarly, for $m = 2$ equations (58d) will give

$$\int_0^1 (B_1 + R^2 B_2)R\,\mathrm{d}R = 0, \tag{108}$$

$$\frac{B_1(0, t)}{2} + \tfrac{1}{4}B_2(0, t) = 0. \tag{109}$$

Using the expression for B_2 from equation (32) and differentiating with respect to time we get

$$\dot{B}_1(0, t) = \frac{1}{2} - \left[\frac{4\dot{A}''(A')^3 - 12(A')^2\dot{A}'A''}{16(A')^6}\right]. \tag{110}$$

The following equations determine completely the initial-boundary-value problem on B_1.

$$\ddot{B}_1 = B_1''\left[\frac{b}{E}\left(1+\frac{2}{(A')^3}\right)\right] - \frac{6b}{E}\frac{A''}{(A')^4} + \eta(Z, t), \tag{111}$$

$$\dot{B}_1(0,t) = -\frac{1}{2}\frac{4\dot{A}''(A')^3 - 12(A')^2\dot{A}'A''}{16(A')^6}\bigg|_{Z=0}, \tag{112a}$$

$$B_1'(Z,0) = \dot{B}_1(Z,0) = 0. \tag{112b}$$

Now let us call $v = \dot{A}$, $s = A'$. It was seen from the treatment of the traction boundary-value problem that if $v(0,t) = V(t)$ and $s(0,t) = n(t)$ are known, then the solution is given by

$$v(Z,t) = V(\alpha), \qquad s(Z,t) = n(\alpha),$$

where

$$\alpha = t - \frac{Z}{c(s)}.$$

In the displacement boundary-value problem $V(t)$ is already prescribed. Using equation (86) at $Z = 0$ we get

$$V(t) = -\int_1^{\tilde{s}} c(\xi)\,\mathrm{d}\xi, \tag{113}$$

where $\tilde{s} = s(0,t)$. Even though the integration in equation (113) cannot be done analytically for the neo-Hookean material, one can obtain $\tilde{s} = s(0,t)$ in a tabulated form by numerical integration. At any point $(0,t^*)$ the slope $c(\tilde{s}(t^*))$ of the α characteristic is known. The values of v and s at $(0,t^*)$ are carried with no change along the characteristic issuing from $(0,t^*)$ with slope $c(\tilde{s}(t^*))$.

For the solution of the second-order effect it is seen from equation (111) that one needs the higher derivatives of $s = A'$, such as s', s'', s''', $\dot{s}'$. To see how these can be obtained let us define

$$-\int_1^{\tilde{s}} c(\xi)\,\mathrm{d}\xi = l(\tilde{s}). \tag{114}$$

Then from equation (113) we have

$$V(t) = l(\tilde{s}).$$

This equation is equivalent to equation (76) with the difference that we possess $l(\tilde{s})$ only in a tabulated form. Again denote:

$$l(\tilde{s}) = V(t) = \zeta,$$

then

$$\tilde{s} = l^{-1}(\zeta) = w(\zeta) = w(V(t)) = n(t).$$

The problem is now completely equivalent to the traction boundary value problem in Section 3.c.2 of [1] with $l(\tilde{s})$ corresponding to $f(\tilde{s})$, ζ corresponding to k, $w(\zeta)$ corresponding to $g(k)$ and $n(t)$ corresponding to $h(t)$. From equation (88) with our new variables we get

$$s' = \frac{\partial n}{\partial \alpha}\alpha'\frac{1}{1-(\partial n/\partial\alpha)\alpha_{,s}},$$

where

$$\frac{\partial n}{\partial \alpha} = \frac{1}{\partial l/\partial \tilde{s}|_{\tilde{s}=\tilde{s}*}}\frac{\partial V}{\partial t}\bigg|_{t=\alpha},$$

where s^* is as previously defined. In the above equation α', $\alpha_{,s}$ and $\partial V/\partial t|_{t=\alpha}$ are readily obtained. Also from equation (113) it is seen that,

$$\frac{\partial l}{\partial \tilde{s}} = -c(\tilde{s}).$$

Higher derivatives of s will involve $\partial^2 l/\partial \tilde{s}^2$, and $\partial^3 l/\partial \tilde{s}^3$ [equations (94), (95)]. These are

$$\frac{\partial^2 l}{\partial \tilde{s}^2} = -\frac{\partial c}{\partial \tilde{s}}, \qquad \frac{\partial l^3}{\partial \tilde{s}^3} = -\frac{\partial^2 c}{\partial \tilde{s}^2}.$$

The displacement boundary-value problem up to $0(\varepsilon^2)$ has thus been solved.

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Абстракт—Асимптотический метод для задачи распространения волны в стержне применяется для нелинейного упругого материала, удовлетворящего нео-гуковому конститутивному закону. Описывается поле перемещений рядами возмущения в малом радиусе стержня. Выражение первого порядка соответствуе элементарной теории, но последующие члены определяют горизонтальную инерцию, сдвиг и эффекты высших порядков. Получаются решения первого и второго порядков для некоторых значений параметра, который характеризует нелинейность конститутивного закона.